Imbalance effects in the Lucas model: An analytical exploration*

Raouf Boucekkine† Ramón Ruiz-Tamarit‡

Abstract

In this note, we use a technique analogous to Xie's method (1994) to solve analytically the Lucas model with externality in a specific parametric case. In particular, we characterize the shape of imbalance effects in this model. Our results are entirely consistent with the findings of the related computational literature. Moreover, our analytical investigation tends to show that these findings are robust to the presence of the Lucas externality as long as a unique equilibrium path exists.

Keywords: Imbalance effects, Lucas model, Externality, Analytical solution.

JEL classification: C61, C62, O41.

---

*The authors thank David de la Croix for an early assessment of this note. Boucekkine acknowledges the support of the Belgian research programmes "Poles d'Attraction inter-universitaires" PAI P5/21, and "Action de Recherches Concertée" ARC 03/08-302. Ruiz-Tamarit acknowledges the support of the Belgian research program ARC 03/08-302, the financial support from the Spanish CICYT, Project SEC2000-0260, and the Grant PR2003-0107 from the Secretaría de Estado de Educación y Universidades, Spanish MECD.

†Corresponding author. IRES and CORE, Université catholique de Louvain, Place Montesquieu, 3, B-1348 Louvain-la-Neuve (Belgique). boucekkine@ires.ucl.ac.be or boucekkine@core.ucl.ac.be

‡University of Valencia, Spain, and IRES, Belgium. ramon.ruiz@uv.es

1 Introduction

A key issue in growth theory is the analysis of human capital accumulation and its interaction with physical capital deepening during the development process. One important line of research in this area is the so-called *imbalance effects*, namely the relationship between the growth rate of GDP and the ratio physical to human capital. A comprehensive and suggestive analysis of these imbalance effects can be found in Barro and Sala-i-Martin (1995), chapter 5. The main outcomes of the analysis can be summarized as follows. First of all, within a one-sector model in which physical and human capital are produced by the same technology, and provided gross investments in both capital forms are non-negative, the growth rate of output is a U-shaped function of the ratio physical to human capital. This means that shortfalls of human capital will have roughly the same growth-enhancing effects as shortfalls of physical capital, which is most doubtful. As mentioned by Barro and Sala-i-Martin (1995), there is little evidence that a shortfall of human capital, as provoked for example by an epidemic, has such a short term effect. Therefore, for a more reliable study of the imbalance effects, there is a need to refine the one-sector model (for example by adding investments' adjustment costs, larger in the case of human capital) or to move to multi-sectoral models.

A natural multi-sectoral set-up to deal properly with imbalance effects is the Lucas model (1988) (see also Lucas, 1993). Mulligan and Sala-i-Martin (1993) and Barro and Sala-i-Martin (1995), chapter 5, have conducted such an analysis, using mainly numerical simulations and/or qualitative reasoning on the Lucas model without externality. They conclude that, in contrast to the one-sector set-up, this two-sector model does not give rise to the U-shaped imbalance effects.[1] The rational behind this finding is rather simple: as the education sector is by construction more intensive in human capital, its operation cost is larger in case of a shortfall of human capital because of the induced higher wage. This motivates people to allocate human capital to the final good sector, rather to the education sector.

Nonetheless, the (mainly) computational approach followed in the above mentioned literature has led certain authors to question its contributions. For example, Xie (1994) describes the transitional dynamics and stability anal-

[1] Naturally, for a proper comparison between the two cases, one has to consider a broad measure of output in the multi-sector model. We will come back to this point in our analytical developments.

ysis in Mulligan and Sala-i-Martin (1993) as non-transparent. More broadly speaking, there is an obvious need to complement this computational and qualitative literature with full-fledged theoretical proofs, and indeed, some few contributions have came out taking exactly this approach. In addition to Xie (1994), Caballé and Santos (1993) and Benhabib and Perli (1994) are by far the most comprehensive and rigorous. While Xie proposes an analytical solution of the Lucas model in a particular parametric case, the two other contributions provide a deep local stability analysis in some general cases.[2]

In this paper, we use a technique analogous to Xie's method to solve analytically the Lucas model with externality in a specific parametric case. In particular, we characterize the shape of imbalance effects in this model. To this end, we fully determine the solution paths of all variables in level in the case where uniqueness of equilibrium paths is ensured, a case unexplored by Xie (1994).[3] Our results are entirely consistent with the findings of Mulligan and Sala-Martin (1993) and Barro and Sala-i-Martin (1995), chapter 5. Moreover, our analytical investigation tends to show that their findings are robust to the presence of the Lucas externality, under certain conditions.

The paper is organized as follows. Section 2 recalls Lucas model and identifies the parametric case where it admits an analytical solution. In this section, we also identify the case where the (analytical) equilibrium solution paths are unique. Section 3 completes the resolution of the model in levels and studies the resulting shape of the imbalance effects. Section 4 concludes.

2 The model

This section recalls the structure of the Lucas model in its competitive equilibrium version, and identifies the parametric case allowing for an analytical solution. It finally establishes the conditions on the parameters of the model in the analytical case under which there exists a unique equilibrium path for each variable. We shall refer to the latter case as the analytical uniqueness case of the Lucas model.

[2]It should be noted that Benhabib and Perli consider the Lucas model with externalities while Caballé and Santos do not.

[3]Xie explores the multiplicity case in order to study overtaking and catching-up.

2.1 The Lucas model

Consider the Lucas (1988) two-sector endogenous growth model, with a production externality in the final good sector associated with the human capital accumulation as studied in Xie (1994) and Benhabib and Perli (1994). The economy is closed, competitive, and populated with many identical, rational agents, who have to choose the controls $c(t)$ and $u(t)$, $\forall t \geq 0$, so as to maximize the following objective function

$$\int_0^\infty \frac{c(t)^{1-\sigma} - 1}{1-\sigma} \, N(t) \, e^{-\rho t} \, dt \tag{1}$$

subject to

$$\overset{\bullet}{K}(t) = Y(t) - N(t)\, c(t) = AK(t)^\beta [u(t)N(t)h(t)]^{1-\beta} h_a(t)^\gamma - N(t)c(t)$$

$$\overset{\bullet}{h}(t) = \delta[1 - u(t)]h(t)$$

where $K(0) = K_0 > 0$ and $h(0) = h_0 > 0$ are given. Here $c(t)$ is the stream of real per capita consumption of a single good. The instantaneous utility function is a CRRA function where σ represents the inverse of the intertemporal elasticity of substitution. The population at time t is $N(t)$, which is assumed to grow at a constant exogenously given rate λ. The constant ρ is the rate of time preference, which is assumed $\rho > \lambda$. In this model $h(t)$ is the human capital level, or the skill level, of a representative worker while $u(t)$ is the fraction of non-leisure time devoted to goods production. The output in the consumption good sector, $Y(t)$, which may be allocated to consumption or to physical capital accumulation depends on the capital stock, $K(t)$, the effective work force, $u(t)N(t)h(t)$, and the average skill level of workers, $h_a(t)$. Parameter β is the elasticity of output with respect to physical capital, and γ is positive and intended to capture the external effects of human capital.
In problem (1) the representative optimizing agent takes $h_a(t)$ as given and, consequently, the competitive solution will be different from the socially optimal allocation. The efficiency parameter A represents the constant technological level in the goods sector of this economy. It is assumed that the growth of human capital does not depend on the physical capital stock, but depends on the effort devoted to the accumulation of human capital, $1-u(t)$, as well as on the achieved human capital stock. The efficiency parameter δ represents the constant technological level in the educational sector. Technology in goods sector shows constant returns to scale over private internal

factors. Technology in educational sector is linear. For the sake of simplicity, it is assumed that there is no physical nor human capital depreciation. The analytical solution technique used in this paper still applies in the presence of nonzero depreciation rates.

The current value Hamiltonian associated with the previous intertemporal optimization problem is

$$H^c(K, h, \theta_1, \theta_2, c, u; A, \sigma, \beta, \gamma, \delta, \{N(t), h_a(t) : t \geq 0\}) =$$

$$= \frac{c^{1-\sigma} - 1}{1-\sigma} N + \theta_1 \left[AK^{\beta}(uNh)^{1-\beta} h_a^{\gamma} - Nc \right] + \theta_2 \delta (1-u) h, \tag{2}$$

where θ_1 and θ_2 are the co-state variables for K and h, respectively. The term h_a, as we have seen, is taken as given in order to calculate the competitive equilibrium. Then, the necessary first order conditions, under the equilibrium condition $h_a = h$ implying that all workers are being treated identically, are

$$c^{-\sigma} = \theta_1 \tag{3}$$

$$\theta_1 (1-\beta) AK^{\beta}(uNh)^{-\beta} Nh^{1+\gamma} = \theta_2 \delta h, \tag{4}$$

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \theta_1 \beta AK^{\beta-1}(uNh)^{1-\beta} h^{\gamma}, \tag{5}$$

$$\overset{\bullet}{\theta}_2 = \rho\theta_2 - \theta_1 (1-\beta) AK^{\beta}(uN)^{1-\beta} h^{-\beta+\gamma} - \theta_2 \delta (1-u), \tag{6}$$

$$\overset{\bullet}{K} = AK^{\beta}(uNh)^{1-\beta} h^{\gamma} - Nc, \tag{7}$$

$$\overset{\bullet}{h} = \delta (1-u) h. \tag{8}$$

As boundary conditions we have the initial conditions K_0 and h_0 and the transversality conditions

$$\lim_{t\to\infty} \theta_1 K \exp\{-\rho t\} = 0, \tag{9}$$

$$\lim_{t\to\infty} \theta_2 h \exp\{-\rho t\} = 0. \tag{10}$$

Finally, notice that since θ_1 (Resp. θ_2) can be viewed as the shadow price of physical capital, or of the physical good (Resp. human capital), the aggregate output of the economy, say Q, can be written as:

$$Q = Y + \frac{\theta_2}{\theta_1} N \overset{\bullet}{h} . \tag{11}$$

Aggregate output is an important variable in the study of imbalance effects, as rightly explained in Barro and Sala-i-Martin (1995) chapter 5. We come back to this variable in Section 3.
This completes the Lucas model. On the margin, according to (3), goods must be equally valuable in its two uses: consumption and physical capital accumulation; according to (4), time must be equally valuable in its two uses: production and human capital accumulation. Moreover, (5) and (6) are the usual intertemporal efficiency conditions for physical and human capital. Equations (7) and (8), in turn, represent their respective accumulation processes.

We now turn to the identification of the parametric case where the model admits a unique analytical solution (referred to hereafter as the "analytical uniqueness case").

2.2 Identification of the analytical uniqueness case

As in Benhabib and Perli (1994), we assume a constant normalized population for simplification. This corresponds to $\lambda = 0$ and $N(0) = 1$. Then, from (3) and (4) we get the control functions

$$c = \theta_1^{-\frac{1}{\sigma}}, \tag{12}$$

$$u = \left(\frac{(1-\beta)\,A}{\delta}\right)^{\frac{1}{\beta}} \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\beta}} h^{\frac{\gamma}{\beta}-1} k. \tag{13}$$

Substituting these expressions in (5)-(8) we obtain the non-linear dynamic system

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \xi\theta_1^{\frac{1}{\beta}}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)} h^{\frac{\gamma}{\beta}} \tag{14}$$

$$\overset{\bullet}{\theta}_2 = -\left(\delta - \rho\right)\theta_2 \tag{15}$$

$$\overset{\bullet}{k} = \frac{\xi}{\beta}\theta_1^{\frac{1}{\beta}-1}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)} k\, h^{\frac{\gamma}{\beta}} - \theta_1^{-\frac{1}{\sigma}} \tag{16}$$

$$\overset{\bullet}{h} = \delta h - \left(\frac{1-\beta}{\beta}\right)\xi\theta_1^{\frac{1}{\beta}}\theta_2^{-\frac{1}{\beta}} k h^{\frac{\gamma}{\beta}}, \tag{17}$$

where k represents the aggregate as well as the per capita physical capital stock, and $\xi \equiv \frac{\beta\delta}{1-\beta}\left(\frac{(1-\beta)A}{\delta}\right)^{\frac{1}{\beta}} > 0$. These equations, together with the initial conditions k_0 and h_0, and the transversality conditions (9) and (10), make the Lucas competitive equilibrium dynamics completely determined over time.

Now let us try to identify an analytical case. At first glance, one can see from (15) that the multiplier θ_2 has a constant growth rate, $-(\delta-\rho)$. Hence,

$$\theta_2 = \theta_2(0)\exp\left\{-(\delta-\rho)\,t\right\}, \tag{18}$$

where $\theta_2(0)$ has to be determined.

Consider now the instrumental variable x defined as[4]

$$x \equiv \theta_1^{\frac{1}{\sigma}} k. \tag{19}$$

By totally differentiating and substituting from (14) and (16) we get

$$\overset{\bullet}{x} = \frac{1}{\sigma}\frac{\overset{\bullet}{\theta_1}}{\theta_1}x + \frac{\overset{\bullet}{k}}{k}x = \frac{\rho}{\sigma}x - \frac{\xi}{\sigma}\theta_1^{\frac{1}{\beta}-1}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}h^{\frac{\gamma}{\beta}}x + \frac{\xi}{\beta}\theta_1^{\frac{1}{\beta}-1}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}h^{\frac{\gamma}{\beta}}x - \frac{x}{\theta_1^{\frac{1}{\sigma}}k},$$

which cannot be solved analytically in the general case. However, under the assumption $\sigma=\beta$, the equation just above transforms into the following non homogeneous first-order first-degree linear differential equation with constant coefficients

$$\overset{\bullet}{x} = \frac{\rho}{\sigma}x - 1. \tag{20}$$

Given k_0 and a certain initial value $\theta_1(0)$, for the moment unknown, we can generate an initial condition for x, namely $x(0) = \theta_1^{\frac{1}{\sigma}}(0)k_0$. Then, a particular solution to (20) is

$$x = \frac{\sigma}{\rho} + \left[x(0) - \frac{\sigma}{\rho}\right]\exp\left\{\frac{\rho}{\sigma}t\right\}. \tag{21}$$

As in Xie (1994), the parametric case $\sigma=\beta$ offers the opportunity to study analytically the Lucas model. Indeed, we can already go further in the analysis of the dynamics of the auxiliary variable x, and establish the following asymptotic property for the latter variable:

Proposition 1 *Along any equilibrium path, x remains constant at the stationary value $x=\frac{\sigma}{\rho}$.*

[4] This corresponds to the inverse of the ratio consumption-physical capital stock considered in Xie (1994).

Proof. From (19) and (21), under the assumption $\sigma = \beta$, we get

$$\theta_1 k = x\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} = \frac{\sigma}{\rho}\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} + \left[x(0) - \frac{\sigma}{\rho}\right]\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\left\{\frac{\rho}{\sigma}t\right\}.$$

Then, (9) may be written as

$$\lim_{t\to\infty} \theta_1 k \exp\{-\rho t\} = \lim_{t\to\infty} \frac{\sigma\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\}}{\rho}$$
$$+ \lim_{t\to\infty} \left[x(0) - \frac{\sigma}{\rho}\right]\theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\left\{\rho\left(\frac{1-\beta}{\beta}\right)t\right\} = 0. \quad (22)$$

Given that x is always non-negative, the transversality condition imposes as a necessary but not sufficient condition that $\lim_{t\to\infty} \theta_1^{-\left(\frac{1-\beta}{\beta}\right)} \exp\{-\rho t\} = 0$. Consequently, looking at the second right-hand term of (22), we realize that the transversality condition also imposes the constraint $x(0) = \frac{\sigma}{\rho}$, from which we deduce the stationarity of x simply by substituting in (21). This is the unique non-explosive solution trajectory for x, which is a constant value given by its initial condition. This result implies a particular and well-defined initial value for θ_1

$$\theta_1(0) = \left(\frac{\sigma}{\rho}\frac{1}{k_0}\right)^{\sigma}, \quad (23)$$

when $\sigma = \beta$. □

It is then quite straightforward to identify the analytical uniqueness case as announced early in this section, by solving sequentially the system (14)-(17). Hereafter, we state a proposition on human capital dynamics in order to make clear the three possibilities that arise even in our analytical case $\sigma = \beta$: multiplicity of equilibrium paths, uniqueness and non-existence.

Proposition 2 *Under the competitive equilibrium conditions,*

i) if $\gamma > \beta$ and $\delta(1+\gamma-\beta) - \rho > 0$ then there exist a continuum of equilibrium paths for h starting from h_0. These paths may be characterized by the multiplicity of initial values $\theta_2(0) = (1+\epsilon)\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} h_0^{\gamma-\beta}$, where $\epsilon \gtreqless 0$ is indeterminate.

ii) if $\gamma > \beta$ and $\delta(1+\gamma-\beta) - \rho \leqslant 0$ then it does not exist any equilibrium path for h starting from h_0.

iii) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho \geqslant 0$ then it does not exist any equilibrium path for h starting from h_0.

iv) if $\gamma < \beta$ and $\delta(1+\gamma-\beta)-\rho < 0$ then there exist a unique equilibrium path for h starting from h_0. This unique path may be characterized by the initial value $\theta_2(0) = \left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta} h_0^{-(\beta-\gamma)}$, given $\epsilon = 0$.

Proof. Using Proposition 1 and (18), one can rewrite (17) as follows

$$\overset{\bullet}{h} = \delta h - \psi_1 h^{\frac{\gamma}{\beta}}, \tag{24}$$

where $\psi_1 = \left(\frac{1-\beta}{\beta}\right)\xi\theta_2^{-\frac{1}{\beta}}(0)\frac{\sigma}{\rho}\exp\left\{\frac{\delta-\rho}{\beta}t\right\}$. Equation (24) may be solved in two steps using Bernoulli's method, which leads to the general solution

$$h = \left\{\left[h_0^{\frac{\beta-\gamma}{\beta}} + W_1\right]\exp\left\{\frac{\delta(\beta-\gamma)}{\beta}t\right\} - W_1\exp\left\{\frac{\delta-\rho}{\beta}t\right\}\right\}^{\frac{\beta}{\beta-\gamma}}, \tag{25}$$

where

$$W_1 = -\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\theta_2^{-\frac{1}{\beta}}(0)\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}.$$

The transversality condition (10), in turn, may be written as

$$0 = \lim_{t\to\infty}\left[(\theta_2(0)h_0)^{\frac{\beta-\gamma}{\beta}} - \frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\theta_2^{-\frac{1+\gamma-\beta}{\beta}}(0)\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right.$$
$$\left.+\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\theta_2^{-\frac{1+\gamma-\beta}{\beta}}(0)\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\exp\left\{\frac{\delta(1+\gamma-\beta)-\rho}{\beta}t\right\}\right]^{\frac{\beta}{\beta-\gamma}}, \tag{26}$$

and the different cases in Proposition 2 arise immediately. □

Remark 1: Notice that multiplicity arises if the externality parameter γ is high enough. In contrast, for uniqueness to arise, the externality should be low enough. This may happen *a fortiori* when there is no externality like in the Lucas-Uzawa model studied by Barro and Sala-i-Martin (1995), chapter 5. However, even in that case, existence is not granted as one can deduce from property iii). For example, for fixed β and ρ, there is no equilibrium path if the technology level in the education sector is large enough.

Remark 2: Our analysis differ notably from Benhabib and Perli (1994) and Caballé and Santos (1993) in that it is not qualitative and local. Moreover, our solution method is able to produce the solution paths for all the original variables (in level), and does not require any dimension reduction as in Benhabib and Perli (1994). Of course, in contrast to the latter contributions, we have to restrict our study to the case $\sigma = \beta$ so as to produce analytical solutions.

Remark 3: Our approach follows the original idea of Xie (1994), notably in the initial phase of the identification of the analytical case. However, there are some minor differences in the early algebraic treatment. Xie (1994) considers the same auxiliary variable x but establishes his main existence and multiplicity results on variable u (Theorem 1, page 101) , instead of h in our case.[5] More importantly, since this author is interested in overtaking and catching-up *via* the multiplicity argument, he restricts his analysis to the case i) and only solves for (detrended) physical and human capital dynamics. We are interested in the shape of imbalance effects as typically studied in Mulligan and Sala-i-Martin (1993). We therefore focus on the uniqueness case iv) and solve for output and aggregate output, Y and Q, respectively, which requires **solving for all the variables of the model in level**. We do this in the next section.

3 The shape of the imbalance effects in the analytical uniqueness case

As argued above, we hereafter focus on the parametric case $\gamma < \beta$ and $\delta (1 + \gamma - \beta) - \rho < 0$.

3.1 Solving the model

We first show that neither human capital nor education time has transitional dynamics in the uniqueness case, contrary to the multiplicity case explored by Xie (1994).

[5] Which implies, as we will see in the next section, a further restriction on the parameters from the condition $u < 1$.

Proposition 3 *If $\gamma < \beta$ and $\delta\,(1+\gamma-\beta) - \rho < 0$, the human capital stock $h(t)$ grows at a constant rate, $\bar{g}_h = \frac{\delta-\rho}{\beta-\gamma} > 0$, if and only if $\delta > \rho$. Therefore, the equilibrium education time is constant, equal to $\bar{u} = -\frac{\delta(1+\gamma-\beta)-\rho}{\delta(\beta-\gamma)}$.*

Proof. Substituting $\theta_2(0)$ given in Proposition 2, iv) in equation (25) we get the unique competitive solution path for human capital

$$h = \bar{h} = h_0 \exp\left\{\frac{\delta-\rho}{\beta-\gamma}t\right\}. \tag{27}$$

Along this path h is positive and grows at a positive constant rate, whenever $\delta > \rho$. Then, the constancy of equilibrium investment time directly derives from (8). □

The next proposition closes the analytical resolution of the model in the uniqueness case.

Proposition 4 *Assume that $\gamma < \beta$, and $\frac{\gamma}{\beta}\ \rho < \delta\,(1+\gamma-\beta) < \rho < \delta$. Then,*

i) the unique positive equilibrium trajectory for k starting from k_0, displays transitional dynamics and approaches asymptotically the unique balanced growth path with the rate, $\bar{g}_k = \frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right) > 0$,

ii) the unique positive equilibrium trajectory for c starts from $c(0) = \frac{\rho}{\beta}k_0$; it shows transitional dynamics and approaches asymptotically the unique balanced growth path with the rate $\bar{g}_c = \bar{g}_k$.

Proof. Substituting the already available solutions paths of $h(t)$ and $\theta_2(t)$ in (14), we get

$$\overset{\bullet}{\theta}_1 = \rho\theta_1 - \psi_2\theta_1^{\frac{1}{\beta}}, \tag{28}$$

where $\psi_2 = \xi\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{\beta-1} h_0^{1+\gamma-\beta}\exp\left\{\frac{\delta-\rho}{\beta-\gamma}\left(1+\gamma-\beta\right)t\right\}$. Equation (28) may be solved as before applying Bernoulli's method, which leads to the solution

$$\theta_1 = \left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} + C_0^0 h_0^{1+\gamma-\beta} I_0(t)\right]^{\frac{-\beta}{1-\beta}} \exp\left\{\rho t\right\},$$

where $C_0^0 = \frac{\left(\frac{1-\beta}{\beta}\right)\xi}{\left(\frac{\left(\frac{\gamma-\beta}{\beta}\right)(1-\beta)\xi\frac{\sigma}{\rho}}{\delta(1+\gamma-\beta)-\rho}\right)^{1-\beta}} > 0$, and $I_0(t) = \frac{(\gamma-\beta)\beta\left(1-\exp\left\{-\frac{\delta\beta(1+\gamma-\beta)-\gamma\rho}{(\gamma-\beta)\beta}t\right\}\right)}{\delta\beta(1+\gamma-\beta)-\gamma\rho}$.

After some substitutions and rearranging terms we get

$$\theta_1 = [\left\{\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} - \frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right\}\exp\left\{-\frac{(1-\beta)\rho}{\beta}t\right\}$$
$$+\frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\exp\left\{\frac{(\delta-\rho)(1+\gamma-\beta)}{(\beta-\gamma)}t\right\}]^{\frac{-\beta}{1-\beta}}. \tag{29}$$

Therefore, given (29) and the prevailing set of parameter constraints, the transversality condition (22) will be always met with no additional constraint on the parameters. In this case, there exist a unique equilibrium path for θ_1, starting from $\theta_1(0)$.

Now we are able to find out the physical capital's solution paths. Indeed, using the previous results for θ_1, h and θ_2 we can substitute in (16) getting

$$\overset{\bullet}{k} = \psi_3 k - \psi_4, \tag{30}$$

where $\psi_3 = \frac{\xi}{\beta}\theta_2^{-\left(\frac{1-\beta}{\beta}\right)}\theta_1^{\frac{1}{\beta}-1}h^{\frac{\gamma}{\beta}} = \frac{1}{\beta}\psi_2\theta_1^{\frac{1-\beta}{\beta}}$ and $\psi_4 = \theta_1^{-\frac{1}{\sigma}}$.
The general solution to (30) is of the form

$$k = k_0\exp\left\{\int_0^t \psi_3(s)\,ds\right\} - \int_0^t \psi_4(r)\exp\left\{\int_r^t \psi_3(z)\,dz\right\}\,dr. \tag{31}$$

Explicit integration yields the exact solution for $k(t)$:

$$k = \frac{\beta}{\rho}[\left\{\left(\frac{\rho}{\beta}k_0\right)^{1-\beta} - \frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right\}\exp\left\{-\frac{(1-\beta)\rho}{\beta}t\right\}$$
$$+\frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\exp\left\{\frac{(\delta-\rho)(1+\gamma-\beta)}{(\beta-\gamma)}t\right\}]^{\frac{1}{1-\beta}}. \tag{32}$$

Property i) of Proposition 4 trivially derives from the equation above. The corresponding balanced growth path is:

$$\bar{k} = \frac{\beta}{\rho}\left(\frac{(\beta-\gamma)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta(1+\gamma-\beta)-\gamma\rho}\right)^{\frac{1}{1-\beta}}\exp\left\{\frac{1+\gamma-\beta}{1-\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}. \tag{33}$$

Property ii) of the proposition straightforwardly comes from (12), (19), and Proposition 1, as these equations imply $c = \frac{\rho}{\beta}k$. $\square$

3.2 The shape of the imbalance effects

Now, we use the analytical solution paths computed in section 3.1 to study the shape of imbalance effects. Recall that such an analysis consists in depicting the relationship between the growth rate of aggregate output (or of output in the consumption good sector) and the gap between the values of the ratio $\frac{k}{h}$ in the short run and along the balanced growth paths respectively. We shall denote this ratio, ω, as in Barro and Sala-i-Martin (1995), section 5.2.2, page 183.
Let us start with the output of the consumption good sector. Using Proposition 3 and 4, one can find that

$$y = Ak^{\beta}u^{1-\beta}h^{1+\gamma-\beta} = A\left(\frac{\beta}{\rho}\right)^{\beta} h_0^{1+\gamma-\beta}\left(-\frac{\delta\left(1+\gamma-\beta\right)-\rho}{\delta\left(\beta-\gamma\right)}\right)^{1-\beta}$$

$$\cdot[\left[\left(\frac{\rho}{\beta}k_0\right)^{1-\beta}-\frac{\left(\beta-\gamma\right)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta\left(1+\gamma-\beta\right)-\gamma\rho}\right]\exp\left\{\frac{\left(1-\beta\right)\left(\delta\left(1+\gamma-\beta\right)-\rho\right)}{\left(\beta-\gamma\right)\beta}t\right\}$$

$$+\frac{\left(\beta-\gamma\right)\beta C_0^0 h_0^{1+\gamma-\beta}}{\delta\beta\left(1+\gamma-\beta\right)-\gamma\rho}\exp\left\{\frac{\left(1+\gamma-\beta\right)}{\beta}\left(\frac{\delta-\rho}{\beta-\gamma}\right)t\right\}]^{\frac{\beta}{1-\beta}}. \tag{34}$$

Therefore, in the considered parametric case, $y(t)$ exhibits transition dynamics and converges to a balanced growth path with the same growth rate as physical capital and consumption. Using (27) and (32)-(34), one can write the growth rate of $y(t)$ as follows:

$$\frac{1}{y(t)}\frac{dy(t)}{dt} = \frac{\delta\left(1+\gamma-\beta\right)-\rho}{\beta-\gamma}+\frac{\delta\beta\left(1+\gamma-\beta\right)-\gamma\rho}{\left(1-\beta\right)\left(\beta-\gamma\right)}\left[\frac{\bar{\omega}\ (t)}{\omega(t)}\right]^{1-\beta}. \tag{35}$$

Therefore, the growth rate of $y(t)$ is inversely related to ω. Indeed, one can check that this is also the case for $k(t)$, $c(t)$, and $q(t)$, the aggregate output given by (11) with $N = 1$. Concerning the latter variable, we have: $q = y + \frac{\theta_2}{\theta_1}\ \dot{h} = y + \frac{\theta_2}{\theta_1}\ \delta\ (1-u)\ h$. Substituting the closed-form solutions of h, u, θ_1 and θ_2 into the latter definition, and comparing the resulting expression to $y(t)$ as given by (34), one finds that:

$$q = \left(1+\left(1-\beta\right)\frac{1-\bar{u}}{\bar{u}}\right)\ y, \tag{36}$$

implying that in our parametric case, equilibrium aggregate output is proportional to the equilibrium production in the consumption good sector. It follows that just like y, q is also a decreasing function of ω. This suggests the following lessons and observations in connection with the literature of imbalance effects.

i) First of all, in contrast to the mainly empirical literature of imbalance effects, we are able to provide an analytical illustration of the typical growth patterns (in terms of the physical to human capital ratio) that arise in the Lucas model. In particular, we show that the growth rate of both production of the final good and aggregate output are simple decreasing and convex functions of the ratio $\frac{\omega}{\bar{\omega}}$. This is entirely consistent with the numerical and qualitative work by Mulligan and Sala-i-Martin (1993) and Barro and Sala-i-Martin (1995) chapter 5.[6]

ii) Notice that our results generalize in some way the above mentioned literature. In particular, this literature typically considers the Lucas model without externality ($\gamma = 0$ in our model), or the so-called Lucas-Uzawa model. We show that the main results on imbalance effects hold when the externality is weak enough, precisely when $0 \leq \gamma < \beta = \sigma$. Observe that the ratio $\omega = \frac{k}{h}$ is no longer a constant in the long run as in the related literature. The presence of a nonzero externality makes it increase at a constant rate in the long run. However, this does not break the typical shape of the growth rate of output reported in the imbalance effects literature, suggesting that this shape is robust to a large extent.

iii) Last, we have to stress here that the obtained adjustment to the balanced growth paths is specific in that human capital and education time adjust immediately to their steady state patterns, while physical capital (and therefore output and consumption) do display transitional dynamics. In other words, in our parametric case $\sigma = \beta$, the transition only takes place in the consumption sector. The picture is different in the more realistic parametric cases analyzed in Mulligan and Sala-i-Martin (1993) for example, where education time and thus

[6] See, in particular, Appendix 5B in Barro and Sala-i-Martin (1995). See also footnote 18, page 191.

human capital adjust slowly to their respective steady state patterns. Nonetheless, while we should acknowledge that the case $\sigma = \beta$ is not the most realistic one, the adjustment mainly through the final good allocation across activities and sectors seems to fit better the way the so-called German and Japanese miracles have taken place (for example, according to Barro and Sala-i-Martin, 1995, page 176).

4 Concluding remarks

In this paper, we have provided an analytical support for the computational and/or qualitative literature of imbalance effects. Using a technique analogous to Xie's method (1994), we have: i) solved analytically the Lucas model in a particular parametric case; ii) checked that the obtained closed-form shapes of imbalance effects are entirely consistent with the findings of the related computational literature; iii) and concluded that these findings seem to be robust to the presence of the Lucas externality as long as a unique equilibrium path exists.

Naturally, just like in Xie (1994), our results are obtained in a particular parametric case. Beside being consistent with the computational literature, this case has some undeniable pedagogical virtues (see for example our point iii) in Section 3.2). However, this should not hide the fact that our analytical treatment is very partial. Clearly, analytical methods *à la* Xie are of limited scope, and there is a need to elaborate more general analytical techniques. This task does not seem impossible in the Lucas model case, and we are currently undertaking it.

References

R. Barro and X. Sala-i-Martin, *Economic Growth*, 1995. McGraw-Hill.

J. Benhabib and R. Perli, Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth, *Journal of Economic Theory* **63** (1994), 113–142.

J. Caballé and M. Santos, On Endogenous Growth with Physical and Human Capital, *Journal of Political Economy* **101** (1993), 1042–1067.

R. E. Lucas, Jr., On the Mechanics of Economic Development, *Journal of Monetary Economics* **22** (1988), 3-42.

R. E. Lucas, Jr., Making a Miracle, *Econometrica* **61** (1993), 251-272.

C. B. Mulligan and X. Sala-i-Martín, Transitional Dynamics in Two Sector Models of Endogenous Growth, *Quarterly Journal of Economics* **108** (1993), 739–773.

D. Xie, Divergence in Economic Performance: Transitional Dynamics with Multiple Equilibria, *Journal of Economic Theory* **63** (1994), 97–112.